Solution to Exercise 10

1. (a)

$$f^*(d) = \sup_{x\in\mathbb{R}^N} (\langle d, x\rangle - f(x)) = \sup_{x\in\mathbb{R}^N} \sum_{i=1}^{N} (d_i x_i + \log x_i) = \sum_{i=1}^{N} \sup_x \in \mathbb{R} \, (d_i x_i + \log x_i) \, .$$

If $d_i \geq 0$, $\sup_{x\in\mathbb{R}} (d_i x + \log x) = +\infty$. Fix $d_i < 0$. Let $g(x) = d_i x + \log x$. Since $g'(x) = d_i + \frac{1}{x}$ and $g''(x) = -\frac{1}{x^2} < 0$. When $g'(x) = 0$, $x = -\frac{1}{d_i}$. Therefore, $g(x)$ is attains supremum at $x = -\frac{1}{d_i}$. Moreover, $g\left(-\frac{1}{d_i}\right) = -1 - \log(-d_i)$. Hence,

$$f^*(d) = \begin{cases} -n - \sum_{i=1}^{N} (\log(-d_i)), & d_i < 0 \text{ for each } i = 1, 2, \ldots, N \\ +\infty, \\ \text{otherwise} \end{cases}$$

(b)

$$f^*(d) = \sup_{x\in\mathbb{R}^N} \left(d^T x - \left(x^T A x + b^T x + c\right)\right) = -c - \inf_{x\in\mathbb{R}^N} \left(x^T A x + (b-d)^T x\right).$$

Let $g(x) = x^T A x + (b-d)^T x$. Since $\nabla g(x) = 2Ax + (b-d)$ and $\nabla^2 g(x) = 2A \succeq 0$. When $\nabla g(x) = 0, 2Ax = d - b$. Therefore, when $d - b \in \text{im}(A)$, $x = \frac{1}{2}A^{-1}(d-b)$ minimizes $g(x)$, and the minimizer does not exist otherwise. When the minimizer of $g(x)$ exists, $g\left(\frac{1}{2}A^{-1}(d-b)\right) = -\frac{1}{4}(d-b)^T A^{-1}(d-b)$. Then.

$$f^*(d) = \begin{cases} \frac{1}{4}(d-b)^T A^{-1}(d-b) - c, & \text{when } d - b \in \text{im}(A) \\ +\infty & \text{otherwise} \end{cases}$$

(c) Recall the definition of dual norm

$$\|y\|_* = \sup_{\|x\|\leq 1} x^T y.$$

To evaluate $f^*(y) = \sup_x \left(y^T x - \|x\|\right)$ we distinguish two cases.

If $\|y\|_+ \leq 1$, then (by definition of dual norm)

$$y^T x \leq \|x\| \quad \text{for all } x$$

and equality holds if $x = 0$; therefore $\sup_x \left(y^T x - \|x\|\right) = 0$.

If $\|y\|_* > 1$, there exists an x with $\|x\| \le 1, x^T y > 1$; then

$$f^*(y) \ge y^T(tx) - \|tx\| = t\left(y^T x - \|x\|\right)$$

and right-hand side goes to infinity if $t \to \infty$

Hence,

$$f^*(y) = \begin{cases} 0 & \|y\|_* \le 1 \\ +\infty & \|y\|_* > 1 \end{cases}$$

2. Assuming $x^* > 0$ is a solution such that $\sum_i (x_i^*)^p = 1$, the KKT stationarity condition reads

$$\nabla_x \left\{ \sum_i \frac{a_i}{x_i} + \lambda \left(\sum_i x_i^p - 1 \right) \right\} = 0 \Leftrightarrow \frac{a_i}{x_i^2} = p\lambda x_i^{p-1}$$

whence $x_i = [a_i/(p\lambda)]^{1/(p+1)}$. Since $\sum_i x_i^p$ should be 1, we get

$$x_i^* = \frac{a_i^{1/(p+1)}}{\left(\sum_j a_j^{p/(p+1)}\right)^{1/p}}.$$

This x^* is optimal because the problem is convex and the KKT conditions are satisfied at this point.

3.Since $\lim_{x \to 0+} f(x) = 0$. Then when $a \le 0$, f is lower semicontinuous on $\mathbb{R}_+$. Then f is closed. Assume $a > 0$. Consider the sublevel set $\{x \in \mathbb{R}_+ : f(x) \le 0\}$. Since $f(x) \le 0$ for any $0 < x \le 1$, $f(0) > 0$, and $f(x) > 0$ for any $x > 1$,

$$\{x \in \mathbb{R}_+ : f(x) \le 0\} = (0, 1]$$

Hence f is not closed if $a > 0$.